

Critical Exponents of the Diluted Ising Model between Dimensions 2 and 4

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Within the massive field-theoretic renormalization-group approach the expressions for the β and γ functions of the anisotropic mn -vector model are obtained for general space dimension d in three-loop approximation. Resumming corresponding asymptotic series, critical exponents for the case of the weakly diluted quenched Ising model ($m = 1, n = 0$), as well as estimates for the marginal order parameter component number m_c of the weakly diluted quenched m -vector model, are calculated as functions of d in the region $2 \leq d < 4$. Conclusions concerning the effectiveness of different resummation techniques are drawn.

KEY WORDS: Critical phenomena; diluted spin systems; Ising model; renormalization group.

1. INTRODUCTION

Study of the critical behaviour of the Ising model has several attractions. On the one hand, the Ising-like models are simple enough, which is of a special advantage in the statistical physics. On the other hand, in spite of their simplicity, such models show rich and interesting behaviour at the critical point. Also, the existence of the exact solution for the two-dimensional Ising model often makes it an object for verifying different approximation schemes. All the stated above yielded the high interest devoted to the problem. In particular a great deal of generalization of the model appeared. Among different ways of generalization, much attention has been devoted to the affect of the impurities on the critical behaviour of the Ising-like models as well as to the investigation of critical regimes of the models

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on the lattices of a non-integer dimension (d). There have been devised different realizations of the last stated generalization. For example, one can approach the concept of non-integer dimensionality either by explicit construction of the non-integer dimensional object, which leads to the concept of a fractal,⁽¹⁾ or by formal carrying out an analytic continuation of the function, which by definition depends on a natural value of dimension.

Within the theory of critical phenomena the latter ambiguity was reflected in examining the critical behaviour of the many-particle systems on fractal^(2, 3) or on abstract hypercubic lattices of the non-integer d . There arose a question whether a model on a fractal lattice (being scale invariant) possesses universality as well as a system on a hypercubic lattice (having translation invariance). The problem has been widely studied but still remains open.⁽⁴⁻⁷⁾ Today's point of view states that the usual demand for "strong universality" (in sense of critical properties depending only on symmetry of the order parameter, interaction range and space dimension) seems not to be obeyed by fractal lattice systems, and for them the concept of universality itself should be revised.^(8, 9)

Speaking about the study of Ising-like models on analytically continued hypercubic lattices of non-integer d , one should note a great variety of theoretical approaches devised for these problems. These include: the Wilson-Fisher ϵ -expansion⁽¹⁰⁾ improved by the summation method;⁽¹¹⁾ Kadanoff lower-bound renormalization applied to some special non-integer dimensions;⁽¹²⁾ high-temperature expansion improved by a variation technique;⁽¹³⁾ finite-size scaling method applied to numerical transfer-matrices data;^(14, 15) new perturbation theory based on the physical branch of the solution of the renormalization group equation;⁽¹⁶⁻¹⁹⁾ fixed dimension renormalization group technique^(20, 21) applied directly to arbitrary non-integer d .^(22, 23) Perhaps the first paper devoted to the study of the Ising model in different, however not non-integer dimension, was⁽²⁴⁾ where non-universal properties of the model were discussed. All these approaches, as well as the computer simulations, confirm the correctness of the universality hypothesis also for non-integer d hypercubic lattices and allow to obtain the critical exponents as functions of d with high accuracy.

In spite of the variety of approaches to treat the general non-integer d case^(11-19, 22, 23) the results for the Ising model critical exponents obtained on their basis lay close to each other; the above mentioned analytic continuations may appear actually equivalent. Note, however, that the analytic continuation in $\epsilon = 4 - d$ at general non-integer dimension leads to the fail of the Yang-Lee theorem.⁽²⁵⁾ On the other hand, the study of Ising-like spin systems on non-integer dimension hypercubic lattices cannot be reduced to the fractal lattice systems⁽²⁵⁾ except for the case of vanishing lacunarity limit⁽⁴⁾ and thus is the task of individual interest.

Returning to the study of the critical behaviour at integer d , one should note that the problem becomes more complicated when studying spin systems with a structural disorder. Whereas the case of the annealed disorder is of less interest from the point of view of determining asymptotical values of critical exponents,⁽²⁶⁾ the weak **quenched** disorder has been a subject of intensive study. Here the Harris criterion⁽²⁸⁾ has been devised. It states that if the heat capacity exponent α_{pure} of a pure model is negative, that is the heat capacity has no divergence at the critical point, impurities do not affect the critical behaviour of the model in the sense that critical exponents remain unchanged under dilution. Only in the case $\alpha_{\text{pure}} > 0$, the critical behaviour of the disordered model is governed by a *new set of critical exponents*. As far as for a $3dm$ -vector spin model only the $3d$ Ising model ($m = 1$) is characterized by $\alpha_{\text{pure}} > 0$, it is the Ising model which is of special interest. And because of the triviality of the annealed disorder in the sense mentioned above, the most interesting object for study is just the quenched Ising model. The appearance of a set of new critical exponents for that model at $d = 3$ is confirmed by the experiments,⁽²⁹⁻³¹⁾ renormalization group (RG) calculations,⁽³²⁻⁴¹⁾ Monte-Carlo (MC)⁽⁴²⁻⁴⁶⁾ and MCRG⁽⁴⁷⁾ simulations.

The situation is not so simple for the $2d$ Ising model. Onsager exact solution of the pure model proves the logarithmic divergence of heat capacity, which yields $\alpha_{\text{pure}} = 0$, and allows one, in accordance with the Harris criterion, to classify this case as a marginal one. Most of the theoretical works suggest that the $2d$ Ising model with a quenched disorder has the same critical behaviour as the $2d$ pure Ising model (except for logarithmic corrections)^(39, 40, 48-53) (see also review⁽⁵⁴⁾). This result is corroborated by MC-simulations on two-dimensional lattices⁽⁵⁵⁻⁵⁹⁾ and experiments.^(60, 61)

Deviations from the expected critical exponents, which sometimes are observed during such computations, are explained by a system being not in the asymptotic region (see ref. 59 for recent study). Nevertheless, some authors assert that for the $2d$ Ising model with a quenched disorder a new critical behaviour appears.^(62, 63)

While the undiluted Ising model at non-integer d was a subject of intensive study,⁽¹⁰⁻¹⁹⁾ it is not the case for the diluted Ising model. Only the work⁽³⁶⁾ can be mentioned here, where the model was studied within the Golner-Riedel scaling field⁽⁶⁴⁾ approach. It is worthwhile to note that the ε -expansion technique applied to this model, due to the fact that RG-equations appear to be degenerated on the one loop level, results in $\sqrt{\varepsilon}$ -expansion for the critical exponents.⁽³⁴⁾ The latter is known up to the three-loop order^(65, 66) The equations of the massive field theory at fixed integer d (refs. 20 and 21) first applied to the diluted Ising model at $d = 2, 3$ in refs. 35 and 37 were found to be the most effective method for investigating

this problem. In order to consider an arbitrary non-integer d the Parisi approach^(20, 21) was generalized in ref. 67 where critical behaviour of the model was studied in a two-loop approximation. The aim of the present work, based on the massive field theoretical approach, is to make a more detailed investigation of the critical behaviour of the diluted $O(m)$ -vector model at arbitrary d . Though it is the case $m = 1$ in which we are interested most of all, we consider the RG-equations for any m , which also allow us to study the crossover in the model at any d . We will obtain the RG-equations within the 3-loop approximation and apply to their analysis different resummation procedures in order to find the most reliable one.

The set-up of the article is as follows. In the next section we introduce the model and the notation. Then we describe the RG-procedure adopted here and give the series for the RG-functions of the weakly diluted quenched m -vector model in the three-loop approximation. Being asymptotic, these series are to be resummed. This is done in Section 2 where different ways of resummation are used. Section 3 concludes our study giving results for the quantitative characteristics of the critical behaviour and discussing them. In the Conclusions we give some general comments to the present work. In the Appendix we list some lengthy expressions for the coefficients of the RG-functions in the three-loop approximation.

1. THE MODEL AND THE RG-PROCEDURE

As it is well known, the critical behaviour of the quenched weakly-diluted m -vector model is governed by a Lagrangian with two coupling constants:⁽³⁴⁾

$$\mathcal{L}(\phi) = \int d^d R \left\{ \frac{1}{2} \sum_{\alpha=1}^n [|\nabla \vec{\phi}^\alpha|^2 + m_0^2 |\vec{\phi}^\alpha|^2] + \frac{v_0}{4!} \left(\sum_{\alpha=1}^n |\vec{\phi}^\alpha|^2 \right)^2 + \frac{u_0}{4!} \sum_{\alpha=1}^n (|\vec{\phi}^\alpha|^2)^2 \right\} \quad (1)$$

in replica limit $n \rightarrow 0$. Here any $\vec{\phi}^\alpha$ is a m -component vector $\vec{\phi}^\alpha = (\phi^{\alpha, 1}, \phi^{\alpha, 2}, \dots, \phi^{\alpha, m})$; $u_0 > 0$, $v_0 < 0$ are bare coupling constants; m_0 is bare mass.

As it was already stated above, we adopt here the massive field theory renormalization scheme^(20, 21) in order to extract the critical behaviour governed by (1). We start from the defined by (1) unrenormalized one-particle irreducible vertex functions

$$\Gamma^{(L, N)}(q_1, \dots, q_L; p_1, \dots, p_N; m_0, u_0, v_0; A_0; d) \quad (2)$$

depending on the wave vectors $\{q\}, \{p\}$, bare parameters m_0, u_0, v_0 and the ultraviolet momentum cutoff A_0 . The vertex functions' dependence on the space dimension d is explicitly noted here as well. We impose the renormalization conditions at zero external momenta and non-zero mass (see refs. 68 and 69 for instance) at the limit $A_0 \rightarrow \infty$ for the renormalized functions⁽⁷⁰⁾ $\Gamma_R^{(0,2)}, \Gamma_{R,u}^{(0,4)}, \Gamma_{R,v}^{(0,4)}, \Gamma_R^{(1,2)}$:

$$\Gamma_R^{(0,2)}(p, -p; m, u, v; d)|_{p=0} = m^2 \quad (3)$$

$$\frac{d}{dp^2} \Gamma_R^{(0,2)}(p, -p; m, u, v; d)|_{p=0} = 1 \quad (4)$$

$$\Gamma_{R,u}^{(0,4)}(\{p_i\}; m, u, v; d)|_{\{p_i\}=0} = m^{4-d}u \quad (5)$$

$$\Gamma_{R,v}^{(0,4)}(\{p_i\}; m, u, v; d)|_{\{p_i\}=0} = m^{4-d}v \quad (6)$$

$$\Gamma_{R,u}^{(1,2)}(q; p, -p; m, u, v; d)|_{q=p=0} = 1 \quad (7)$$

with m, u, v being the renormalized mass $m = Z_3 m_1 = Z_3 \Gamma^{(0,2)}(0; m_0, u_0, v_0)$ and couplings $u = m^{d-4} Z_3^2 Z_{1,u}^{-1} u_0, v = m^{d-4} Z_3^2 Z_{1,v}^{-1} v_0$. From these conditions there follow expansions for the renormalized constants for field (Z_3), vertices $u(Z_{1,u}), v(Z_{1,v})$ and ϕ^2 insertion (Z_2). Subsequently, these define the coefficients β, γ entering the corresponding Callan-Symanzik equation:

$$\beta_u(u, v) = \left. \frac{\partial u}{\partial \ln m} \right|_{u_0, v_0} \quad (8)$$

$$\beta_v(u, v) = \left. \frac{\partial v}{\partial \ln m} \right|_{u_0, v_0} \quad (9)$$

$$\gamma_\phi \equiv \gamma_3 = \left. \frac{\partial Z_3}{\partial \ln m} \right|_{u_0, v_0} \quad (10)$$

$$\bar{\gamma}_{\phi^2} \equiv \gamma_2 = - \left. \frac{\partial Z_2}{\partial \ln m} \right|_{u_0, v_0} \quad (11)$$

In the stable fixed point $\{u^*, v^*\}$ to be defined by simultaneous zero of both β -functions:

$$\begin{aligned} \beta_u(u^*, v^*) &= 0 \\ \beta_v(u^*, v^*) &= 0 \end{aligned} \quad (12)$$

the γ_ϕ -function gives the critical exponent η of the pair correlation function:

$$\gamma_\phi(u^*, v^*) = \eta \quad (13)$$

The correlation length critical exponent ν is defined in the stable fixed point by:

$$\bar{\gamma}_{\phi^2}(u^*, v^*) = 2 - \nu^{-1} - \gamma_{\phi}(u^*, v^*) \quad (14)$$

Using familiar scaling relations, one can easily calculate any other critical exponents on the base of η and ν .

Applying the described above procedure, one obtains in the three-loop approximation⁽⁷¹⁾ β - and γ -functions in the form:⁽⁷²⁾

$$\begin{aligned} \beta_u(u, v) = & -(4-d) u \left\{ 1 - u - \frac{12}{mn+8} v + \frac{8}{(m+8)^2} \right. \\ & \times \left[(5m+22) \left(i_1 - \frac{1}{2} \right) + (m+2) i_2 \right] u^2 + \frac{96}{(m+8)(mn+8)} \\ & \times \left[(m+5) \left(i_1 - \frac{1}{2} \right) + \frac{m+2}{6} i_2 \right] uv + \frac{24}{(mn+8)^2} \\ & \left. \times \left[(mn+14) \left(i_1 - \frac{1}{2} \right) + \frac{mn+2}{3} i_2 \right] v^2 + \beta_u^{(3LA)} + \dots \right\} \quad (15) \end{aligned}$$

$$\begin{aligned} \beta_v(u, v) = & -(4-d) v \left\{ 1 - v - \frac{2(m+2)}{m+8} u + \frac{8}{(mn+8)^2} \right. \\ & \times \left[(5mn+22) \left(i_1 - \frac{1}{2} \right) + (mn+2) i_2 \right] v^2 + \frac{96(m+2)}{(m+8)(mn+8)} \\ & \left. \times \left[i_1 - \frac{1}{2} + \frac{i_2}{6} \right] uv + \frac{24(m+2)}{(m+8)^2} \left[i_1 - \frac{1}{2} + \frac{i_2}{3} \right] u^2 + \beta_v^{(3LA)} + \dots \right\} \quad (16) \end{aligned}$$

$$\begin{aligned} \gamma_{\phi}(u, v) = & -2(4-d) \left\{ \left[\frac{2(m+2)}{(m+8)^2} u^2 + \frac{4(m+2)}{(m+8)(mn+8)} uv \right. \right. \\ & \left. \left. + \frac{2(mn+2)}{(mn+8)^2} v^2 \right] i_2 + \gamma_{\phi}^{(3LA)} + \dots \right\} \quad (17) \end{aligned}$$

$$\begin{aligned} \bar{\gamma}_{\phi^2}(u, v) = & (4-d) \left\{ \frac{m+2}{m+8} u + \frac{mn+2}{mn+8} v - 12 \left[\frac{m+2}{(m+8)^2} u^2 \right. \right. \\ & \left. \left. + \frac{2(m+2)}{(m+8)(mn+8)} uv + \frac{mn+2}{(mn+8)^2} v^2 \right] \left(i_1 - \frac{1}{2} \right) + \bar{\gamma}_{\phi^2}^{(3LA)} + \dots \right\} \quad (18) \end{aligned}$$

Here d is the space dimension, m is the order parameter component number, n is the replica index, i_1 and i_2 are dimensionally dependent two-loop integrals. The corresponding coefficients for three-loop parts are listed in the Appendix. The values for the three-loop integrals $i_3 \dots i_8$ which appear in three-loop coefficients for integer $d=2, 3$ are listed in.⁽⁷³⁾ In particular, substituting loop integrals i_1, i_2 as well as i_3, \dots, i_8 in (15)–(18) by their values at $d=3$ we get at $n=0, m=1$ the corresponding functions of the $3d$ weakly diluted Ising model, which in the 3-loop approximation were obtained in ref. 35. At $d=3, m, n$ -arbitrary corresponding expressions coincide with those, obtained for the $3d$ anisotropic mn -vector model in ref. 74. Our idea is to keep the dimensional dependence of the loop integrals and, being based on their numerical values for arbitrary d ,⁽²³⁾ to study the $O(mn)$ -model at arbitrary (non-integer) d as well. But for the reason explained above, the point of main interest here will be the replica limit $n=0$ of the anisotropic mn -vector model, especially the case $m=1$.

Expressions for β - and γ -functions will be the starting point for the qualitative study of the main features of the critical behaviour which will be done in the next section.

2. THE RESUMMATION

As we have already mentioned, the values of the γ -functions in a fixed point (u^*, v^*) lead to the values of the critical exponents η and ν . However, it is well known now that the series for RG-functions are of asymptotic nature^(75–77) and imply the corresponding resummation procedure to extract reliable data on their basis. Let us note, however, that as to our knowledge the asymptotic nature of the series for RG-functions have been proved only for the case of the model with one coupling,⁽⁷⁸⁾ and the application of a resummation procedure to the case of several coupling constants is based rather on general belief than on a proved fact. One of the resummation procedures, which in different modifications is most commonly used in the studies of asymptotic series, is known as the integral Borel transformation.⁽⁷⁹⁾ However, this technique implies explicit knowledge of the general term of a series and thus cannot be applied here, where only truncated sums of the series are known. To get over this obstacle one represents the so-called Borel-Leroy image of the initial sum in the form of a rational approximant and in such a way reconstitutes the general term of the series. The technique which involves a rational approximation and the Borel transformation together, is known as the Padé-Borel resummation technique (in the field-theoretical RG content see refs. 81 and 82 as an example of its application).

Note here that the resummation technique, based on the conformal mapping, which is widely used in the theory of critical phenomena,⁽⁸³⁾ cannot be applied in our case because its application postulates information on the high order behaviour of the series for β - and γ -functions. The latter is still unknown for the theory with the Lagrangian (1).

To summarize up the stated let us write that the Padé-Borel resummation is performed as follows:

- constructing the Borel-Leroy image of the initial sum S of n terms:

$$S = \sum_{i=0}^n a_i x^i \Rightarrow \sum_{i=0}^n \frac{a_i (xt)^i}{\Gamma(i+p+1)} \quad (19)$$

where $\Gamma(x)$ is the Euler's gamma function and p is an arbitrary non-negative number. The special cases $p=0$ and $p=1$ correspond to resumming β -functions without or with prefactors u and v in accordance with the structure of the functions (15)–(16);

- the Borel-Leroy image (19) is extrapolated by a rational approximant $[M/N](xt)$, where by $[M/N]$ one means the quotient of two polynomials; M is the order of the numerator and N is that of the denominator;
- the resummed function S^{res} is obtained in the form:

$$S^{\text{res}} = \int_0^\infty dt \exp(-t) t^p [M/N](xt) \quad (20)$$

In the two variables case only the first step is changed; namely, here we define the Borel-Leroy image as

$$\sum_{0 \leq i+j \leq n} a_{i,j} x^i y^j \Rightarrow \sum_{0 \leq i+j \leq n} \frac{a_{i,j} (xt)^i (yt)^j}{\Gamma(i+j+p+1)} \quad (21)$$

Generalization to the many-variable case is trivial.

Now one can take into account that the second step of the stated scheme can be done in different ways. one can write down various Padé approximants in the variable t to obtain within the three-loop approximation the expressions of the structure $[2/1]$, $[1/2]$ and $[0/3]$. It is also possible to use Chisholm approximants⁽⁸⁴⁾ in the variables u and v , which, generally speaking, in the same number of loops can be of type $[3/1]$, $[2/2]$, $[1/3]$ and $[0/4]$, but the explicit definition of any approximant needs some additional equations now.⁽⁸⁴⁾ The technique, which involves Chisholm approximation together with the integral Borel transformation is referred to as the Chisholm-Borel resummation technique. To be consistent, one would have to apply the all different resummation frameworks in

order to obtain reliable results on their basis and find which of the methods is the most effective. However, strong restriction on the number of choices can be imposed.

First of all, an approximant should be chosen in the form reconstituting the signalternating high-order behaviour of the general term of β - and γ -functions, which was confirmed in the particular case $m = 1$, $n = 2$ and $n = 3$.⁽⁸⁵⁾ The approximant generating a sign-alternating series might be chosen in a form $[M/1]$ with the positive coefficients at the variable t (or u and v). Choosing an approximant with a non-linear denominator, generally speaking, one does not ensure the desired properties. Direct calculations affirm the argumentation: β -functions, resummed with the Padé-Borel and the Chisholm-Borel methods with approximants $[M/N]$, $N > 1$, for $u < 0$, $v > 0$ give the roots which lie far from the expected values which for $d = 3$ are known up to the order of four loops⁽³⁹⁾ and for general d were calculated from the two-loop β -functions.⁽⁶⁷⁾ This is true for any p . The stated results permit us to eliminate from the consideration approximants with a non-linear denominator.

Note as well that choosing representation of the RG-functions (15)–(18) in the form of Padé or Chisholm approximant of type $[M/1]$ might result in the appearance of a pole in the expression. Here we use an analytical continuation of the resulting expressions by evaluating the principal value of the integral. Treating the task in this way one notes that the topological structure of the lines of zeros for the resummed by the Padé-Borel technique β -functions is very different in the region near the solution for the mixed fixed point and strongly irregular when passing through the number of loops. In particular this yields that in the three-loop approximation there exist two solutions close to the expected value of the mixed fixed point. To compare, the results obtained within the frames of the Chisholm-Borel method do not have these faults and are more stable from the point of view of proceeding in number of loops.

So, the results given below are obtained by the Chisholm-Borel method applied to the approximant of type $[3/1]$. In order to determine the form of this approximant completely one must define two additional conditions. The approximants are expected to be symmetric in variables u and v , otherwise the properties of the symmetry related to these variables would depend, except for the properties of the Lagrangian, on the method of calculation. By the substitution $v = 0$ all the equations which describe the critical behaviour of the diluted model are converted into appropriate equations of the pure model. However, if pure model is solved independently, the resummation technique with the application of Padé approximant is used. Thus, Chisholm approximant is to be chosen in such a way that, by putting any of u or v equal to zero, one obtains Padé approximant

for a one-variable case. This also implies a special choice of additional conditions. In the present study amidst all the possible expressions which satisfy the stated demand we choose Chisholm approximant [3/1] by putting coefficients at u^3 and v^3 to be equal to zero.

3. RESULTS

Now we are going to apply the mathematical framework which was discussed in previous sections in order to obtain numerical characteristics of the critical behaviour of the weakly-diluted Ising model in general dimensions. It was noted in the Section 1 that the critical behaviour of the quenched weakly-diluted Ising model is described by the effective Lagrangian (1) in the case $m = 1$ and zero replica limit $n = 0$. Namely, the task in the end comes to obtaining fixed points which are defined by simultaneous zero of the both β -functions. Among all the possible fixed points one is interested only in those in the ranges $u^* > 0$, $v^* \leq 0$ and only in stable ones where the stability means that two eigenvalues b_1, b_2 of the stability matrix $B = \partial \beta_{u_i} / \partial u_j |_{u^*}$, $u_i \equiv \{u, v\}$ are positive or possess positive real parts. The structure of the β -functions (15)–(16) yields the possibility of four solutions for the fixed points. The first two $\{u^* = 0, v^* = 0\}$ and $\{u^* = 0, v^* > 0\}$ in our case at $d < 4$ are out of physical interest, while the second pair which consists of pure $\{u^* > 0, v^* = 0\}$ and mixed $\{u^* > 0, v^* < 0\}$ points, are responsible for two possible critical regimes. The critical behaviour of the diluted model coincides with that of the pure model when the pure fixed point appears to be stable. If the mixed point is stable, the *new* (diluted) critical behaviour of the system takes place. The type of the critical behaviour depends on the number m of the order parameter components and on the dimensionality d : at any d , $2 \leq d < 4$ a system with large enough m is not sensitive to the weak dilution in the sense that asymptotic values of critical exponents do not change; only starting from some marginal value m_c , at $m < m_c$ a mixed fixed point becomes stable and the crossover to the random critical behaviour occurs. The problem of determining m_c as a function of d will be discussed later. Now we would like to state that $m_c \geq 1$ for any d , $2 \leq d < 4$, and thus just the mixed fixed point governs the asymptotic critical behaviour of the diluted Ising model.

If one attempts to find the fixed points from the β -functions (15)–(16) without resummation, there always appears only the Gaussian $\{u^* = 0, v^* = 0\}$ trivial solution; the existence of the rest possible three fixed points depends on the concrete details of the β -functions portions in the braces in expressions (15)–(16). In a $3d$ case it appears that without a resummation the non-trivial mixed fixed point does not exist in one-, two- and four-loop approximations.^(39, 40) It is only the three-loop approximation where all the

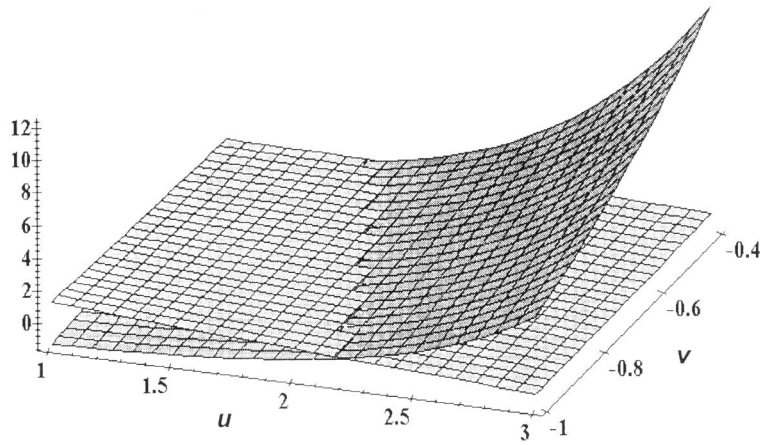


Fig. 1. The non-resummed β -functions in the three-loop approximation; $d = 3, m = 1, n = 0$. The dark surface corresponds to the β_u -function.

four solutions of the set of equations (12) exist.⁽³⁵⁾ In Fig. 1 we show the behaviour of the non-resummed β -functions of the three-dimensional weakly diluted Ising model in the three-loop approximation. Resummed functions are shown in the same approximation in Fig. 2. Note that in this approximation the shape of the functions remains alike in the region of small couplings u and v . Fixed points correspond to the crossing of the lines $\beta_u = 0, \beta_v = 0$ as it is demonstrated in Figs. 3 and 4. The left-hand

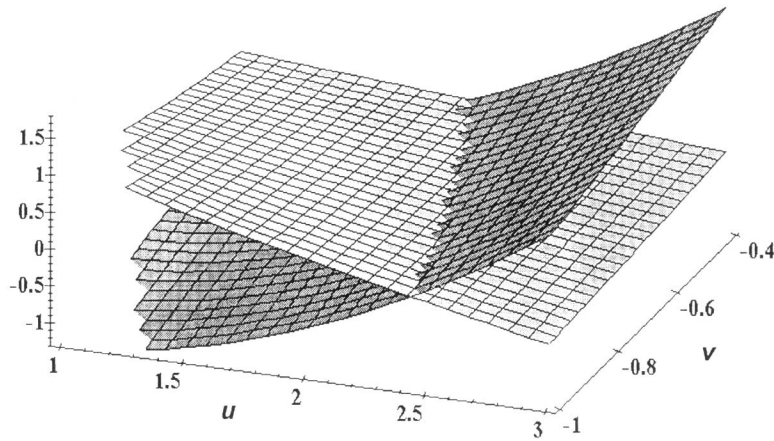


Fig. 2. The Chisholm-Borel resummed β -functions in the three-loop approximation; $d = 3, m = 1, n = 0$. The dark surface corresponds to the β_u -function.

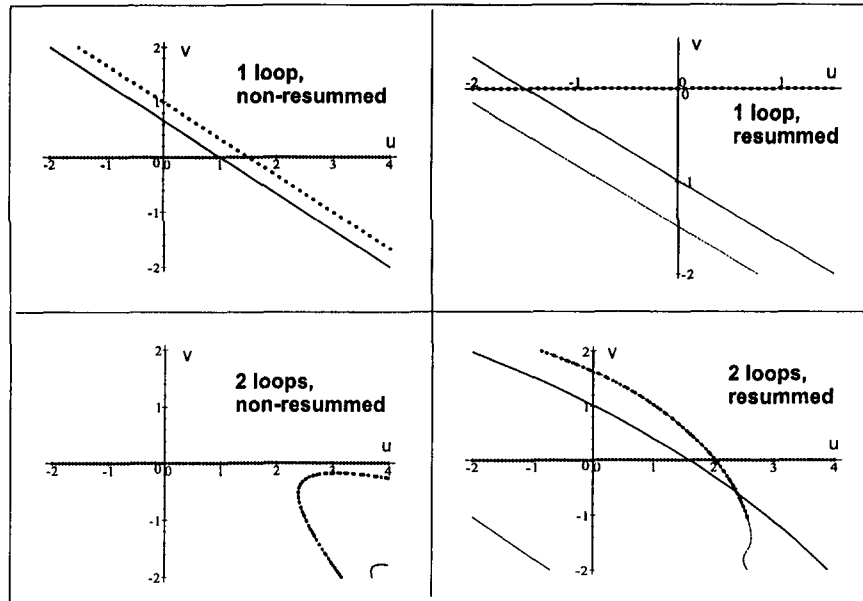


Fig. 3. The lines of zeros of non-resummed (left-hand column) and resummed by the Chisholm-Borel method (right-hand column) β -functions for $m=1$, $n=0$ in different orders of the perturbation theory: one- and two-loop approximations. Circles correspond to $\beta_u=0$, thick lines depict $\beta_v=0$. Thin solid and dashed lines show the roots of the analytically continued functions β_u and β_v respectively. One can see the appearance of the mixed fixed point $u>0$, $v<0$ in the two-loop approximation for the resummed β -functions.

column in Figs. 3 and 4 shows the lines of zeros of non-resummed β -functions in three-dimensions in one-, two-, three- and four-loop (results of refs. 39 and 40) approximations. One can see in the figures that without resummation all non-trivial solutions are obtained only within the three-loop level of the perturbation theory. In the next order all fixed points disappear which is a strong evidence of their accidental origin. At any arbitrary d , $2 \leq d < 4$ the qualitative behaviour of the functions is very similar to that shown in Figs. 3 and 4.

As it has already been mentioned, in order to reestablish the lost pure and mixed points one applies the resummation procedure to β -functions. In the three-dimensional space the result of resummation is illustrated by the right-hand column in Figs. 3 and 4. Here we have used the Chisholm-Borel resummation technique choosing Chisholm approximant in the form discussed in the previous Section with $p=1$ in successive approximation in the number of loops. The icons in the figures which correspond to a one-loop level are the visual proof of the degeneracy of the β -functions in this

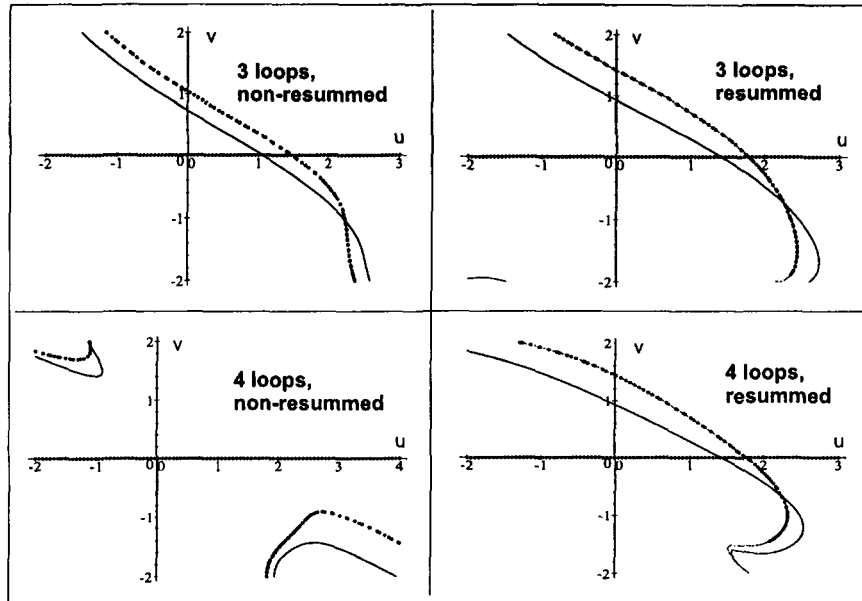


Fig. 4. The lines of zeros of non-resummed (left-hand column) and resummed by the Chisholm-Borel method (right-hand column) β -functions for $m = 1, n = 0$ in three- and four-loop approximations. The notations are the same as in Fig. 3. Close to the mixed fixed point the behavior of the resummed functions remains alike with the increase of the order of approximation. This is not the case for non-resummed functions.

order of the perturbation theory: the plots of root-lines are parallel independently of resummation. The rest three images in the right-hand columns are a good graphic demonstration of the reliability of the Chisholm-Borel resummational method: two-, three- and four-loop pictures are quantitatively similar, the coordinates of the pure and mixed point are close.

The numerical results of our study are given in Table I. Here, the coordinates of the stable mixed fixed point and the values of the critical exponents of the quenched weakly diluted Ising model are listed as functions of d between $d = 2$ and $d = 3.8$. The eigenvalues b_1 and b_2 of the stability matrix are given as well.

It was already noted that the values of γ -functions in a stable point yield the numerical characteristics of the critical behaviour of the model. For example, given the resummed functions γ_ϕ^{Res} and $\bar{\gamma}_{\phi^2}^{\text{Res}}$, the pair of equations

$$\gamma_\phi^{\text{Res}}(u^*, v^*) = \eta \tag{22}$$

$$\bar{\gamma}_{\phi^2}^{\text{Res}}(u^*, v^*) = 2 - v^{-1} - \eta \tag{23}$$

Table 1. The Stable Point Coordinates, Critical Exponents and the Eigenvalues of the Stability Matrix of the Weakly Diluted Ising Model at Arbitrary d^a

| d | u^* | v^* | γ | ν | α | η | b_1 | b_2 |
|-----|--------|---------|----------|-------|----------|--------|---------------------|---------------------|
| 2.0 | 2.0268 | -0.2802 | 1.840 | 0.966 | 0.067 | 0.097 | 0.2176 | 1.5189 |
| 2.1 | 2.0327 | -0.3156 | 1.768 | 0.923 | 0.062 | 0.084 | 0.2373 | 1.4608 |
| 2.2 | 2.0412 | -0.3523 | 1.703 | 0.884 | 0.056 | 0.073 | 0.2562 | 1.4011 |
| 2.3 | 2.0525 | -0.3908 | 1.643 | 0.848 | 0.049 | 0.064 | 0.2742 | 1.3395 |
| 2.4 | 2.0671 | -0.4312 | 1.588 | 0.816 | 0.041 | 0.055 | 0.2913 | 1.2759 |
| 2.5 | 2.0854 | -0.4740 | 1.536 | 0.787 | 0.033 | 0.047 | 0.3074 | 1.2100 |
| 2.6 | 2.1081 | -0.5196 | 1.489 | 0.760 | 0.025 | 0.040 | 0.3226 | 1.1418 |
| 2.7 | 2.1359 | -0.5687 | 1.445 | 0.735 | 0.016 | 0.034 | 0.3370 | 1.0709 |
| 2.8 | 2.1698 | -0.6219 | 1.404 | 0.712 | 0.007 | 0.028 | 0.3505 | 0.9971 |
| 2.9 | 2.2113 | -0.6803 | 1.365 | 0.691 | -0.002 | 0.023 | 0.3635 | 0.9197 |
| 3.0 | 2.2621 | -0.7454 | 1.328 | 0.671 | -0.016 | 0.019 | 0.3764 | 0.8380 |
| 3.1 | 2.3250 | -0.8190 | 1.294 | 0.652 | -0.021 | 0.015 | 0.3905 | 0.7504 |
| 3.2 | 2.4039 | -0.9038 | 1.261 | 0.634 | -0.030 | 0.012 | 0.4095 | 0.6528 |
| 3.3 | 2.5044 | -1.0040 | 1.230 | 0.618 | -0.038 | 0.009 | 0.4653 | 0.5127 |
| 3.4 | 2.6359 | -1.1259 | 1.200 | 0.602 | -0.046 | 0.006 | 0.4436 ^c | 0.4436 ^c |
| 3.5 | 2.8140 | -1.2804 | 1.171 | 0.587 | -0.054 | 0.004 | 0.3946 ^c | 0.3946 ^c |
| 3.6 | 3.0678 | -1.4869 | 1.144 | 0.572 | -0.061 | 0.002 | 0.3411 ^c | 0.3411 ^c |
| 3.7 | 3.4570 | -1.7849 | 1.116 | 0.558 | -0.066 | 0.001 | 0.2822 ^c | 0.2822 ^c |
| 3.8 | 4.0852 | -2.2303 | 1.087 | 0.544 | -0.066 | 0.000 | 0.2136 ^c | 0.2136 ^c |

^a The three-loop approximation (the superscript "c" denotes that real parts of the corresponding eigenvalues are given).

allows us to find the exponents η and ν . All other exponents can be obtained from the familiar scaling laws.

However, one can proceed in a different way. That is, by means of the scaling laws it is possible to reconstitute the expansion in coupling constants of any exponent of interest or of any combination of exponents, and only after that to apply the resummation procedure. If exact calculation were performed the answer would not depend on the sequence of operations. However, this is not the case for the present approximate calculations. We have chosen the scheme of computing where the resummation procedure was applied to the combination $\nu^{-1} - 1 = 1 - \bar{\gamma}_\phi^2 - \gamma_\phi$ and $\gamma^{-1} = (2 - \bar{\gamma}_\phi^2 - \gamma_\phi)/(2 - \gamma_\phi)$. The exponents α , β and η have been calculated on the basis of numerical values of the exponents γ and ν . The resummation scheme appears to be quite insensitive to the choice of the parameter p given by (19), (20). However note, that computations have been performed here, as well as in ref. 67, with $p = 1$.

Comparing our data from Table I for the critical exponents at $d = 2$ with the results for the pure Ising model one can see that the exponent γ

differs from the exact value $7/4$ by the order of 5%, the exponent ν is smaller from the exact value $\nu = 1$ less than by 4%. This confirms the conjecture that the critical behaviour of the weakly diluted quenched Ising model at $d = 2$ within logarithmic correction coincides with that of the pure model (see ref. 54 for review). It is also interesting to compare numbers given in Table I with those obtained for general d within the 2-loop approximation:⁽⁶⁷⁾ all the exponents of the three-loop level lie slightly farther from the expected exact values of Onsager than those of the two-loop approximation. This may be explained by the oscillatory nature of approaching to the exact values depending on the order of the perturbation theory. It is also interesting to note that the two-loop approximation yields better estimates for the heat capacity critical exponent α for all d in the range under consideration. Namely, in accordance with the Harris criterion, the exponent α for the diluted Ising system should remain negative. This picture is confirmed much better by the two-loop approximation where α is negative in the whole range of d , unlike the three-loop level of the perturbation theory, the results of which yield $\alpha > 0$ for $2 \leq d \leq 2.8$.

However, Table I shows that the next (third) order does improve our understanding of the critical behaviour of the model in general dimensions. The results of the two-loop calculations⁽⁶⁷⁾ show that starting from some marginal space dimension the approach to the stable point becomes oscillatory: the eigenvalues b_1 and b_2 turn to be complex possessing positive real parts. This is an artifact of the calculation scheme and therefore it was expected⁽⁶⁷⁾ that by increasing the accuracy of calculations one decreases the region of d which corresponds to the complex eigenvalues. It is really the case. In the three-loop approximation the region of complex b_1, b_2 is bounded from below by $d = 3.3$, whereas in the two-loop approximation⁽⁶⁷⁾ the corresponding value is lower and is equal to $d = 2.9$. Thus, the region of d characterized by the oscillatory approach to the stable fixed point shrinks with the increase of the order of the perturbation theory.

The comparison of the three-dimensional value of ν with the four-loop result⁽⁴⁰⁾ $\nu = 0.6701$ gives the accuracy of 0.05% for our computations (compare with 1% for two-loops, where the value $\nu(d = 3) = 0.678$ was obtained). Thus, it may be stated that the general accuracy of calculations decreases when passing from $d = 4$ to $d = 2$ which, in particular, results from the fact that our approach is asymptotically exact at upper critical dimension $d = 4$.

The comparison of the present results with the other data available is provided by Fig. 5. Here, the behaviour of the correlation length critical exponent ν obtained by different methods is demonstrated in general dimensions. The results of the massive field-theoretical scheme are plotted

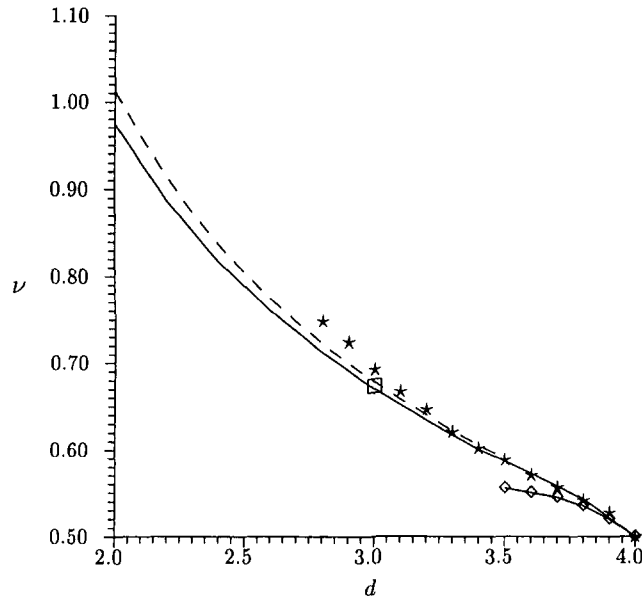


Fig. 5. The correlation length critical exponent ν of the weakly diluted Ising model as a function of the space dimension d . The result of two- (ref. 67) and three-loop (the present paper) approximations are shown by the dashed and the solid lines respectively, the square reflects the number of the four-loop approximation⁽⁴⁰⁾ at $d=3$, stars correspond to work⁽³⁶⁾ and open diamonds refer to the resummed $\sqrt{\varepsilon}$ -expansion.

by solid (three-loop approximation; the present paper) and dashed (two-loop approximation; ref. 67) lines. One can see that the two lines practically coincide far enough from $d=2$, in particular, both lie very close to the most accurate result for $d=3$ (ref. 40) which is shown by the box. The application of the scaling-field method⁽³⁶⁾ yields numbers shown in Fig. 5 by stars. The limit from below ($d=2.8$) of the method applicability is caused by the truncation of the set of scaling-field equations, which was considered in ref. 36. One can also attempt to obtain some results by resumming the $\sqrt{\varepsilon}$ -expansion which is known for the diluted Ising model up to three-loop order^(65,66) and for the exponents ν and η reads:

$$\nu = \frac{1}{2} + \frac{1}{4} \left(\frac{6}{53} \varepsilon \right)^{1/2} + \frac{535 - 756\zeta(3)}{8(53)^\varepsilon} \quad (24)$$

$$\eta = -\frac{\varepsilon}{106} + \left(\frac{6}{53} \right)^{1/2} \frac{9}{(53)^2} (24 + 7\zeta(3)) \varepsilon^{3/2} \quad (25)$$

where $\zeta(3) \approx 1.202$ is Riemann's zeta function. The corresponding results are shown by open diamonds. They were obtained by applying the Padé-Borel resummation scheme to the series of $\sqrt{\varepsilon}$ -expansion (24).^(65,66) The value of ν obtained in such a way is of physical interest only very close to $d=4$. Even in the next orders of the expansions the values of critical exponents are not improved;⁽⁸⁶⁾ this is an evidence of the $\sqrt{\varepsilon}$ -expansion unreliability in tasks like the one under consideration. To compare, one can state that the situation with the applied in the present paper theoretical scheme is contrary to the $\sqrt{\varepsilon}$ -expansion. While the two-loop approximation is valid in ranges $2 \leq d < 3.4$, the next order of the perturbation theory enlarges the upper bound up to $d=3.8$. One can expect that the next steps within the perturbation theory will allow one to obtain the description of the critical behaviour of the model with enough accuracy for any d , $2 \leq d < 4$.

Let us recall now that expressions (15)–(18) for the RG-functions, as well as their three-loop parts listed in the Appendix, allow us to study asymptotic critical properties of the mn -vector model with arbitrary m and n in arbitrary d not only for the case $m=1$, $n=0$. In particular, by keeping m as an arbitrary number and putting $n=0$ one can obtain the numerical estimates for the marginal order parameter component number m_c which divides the diluted (governed by the mixed fixed point) asymptotic critical behaviour from the pure one, when the $O(m)$ -symmetric fixed point remains stable. In accordance with the Harris criterion the case $m=m_c$ corresponds to zero of the heat capacity critical exponent α of the model. One may extract the value of m_c from this condition. However, the above discussed results of the three-loop approximation do not yield enough accuracy for α . Alternatively, the fixed mixed point should coincide with the pure fixed point at $m=m_c$, which in particular means that $v^*(m=m_c)|_{\text{mixed}}=0$. The last condition was chosen as a basis of our calculation. The appropriate numbers of the present three-loop approximation (thick solid line) together with the data of the two-loop approximation (dashed line)⁽⁶⁷⁾ are shown in Fig. 6. The result of ε -expansion $m_c=4-4\varepsilon$ is depicted by the thin solid line. In the three-loop approximation we obtain $m_c=1.40$, $d=2$ and $m_c=2.12$, $d=3$. These values are to be compared with the exact results of Onsager which yield $m_c=1$ at $d=2$, and the theoretical estimate $m_c=1.945 \pm 0.002$.⁽⁸⁷⁾ One can see that the two-loop results are closer to the expected values for both $d=2$ and $d=3$. For a two-dimensional case the two-loop value $m_c=1.19$ (ref. 67) differs from the exact one by 20%, while the three-loop number decreases the accuracy to 40%. The case $d=3$, $m_c > 2$ contradicts the suggestion that the xy -model asymptotic critical behaviour should not change under dilution in three-dimensions. The reason for decreasing the calculation accuracy with increasing the order of the perturbation theory may lie in oscillatory

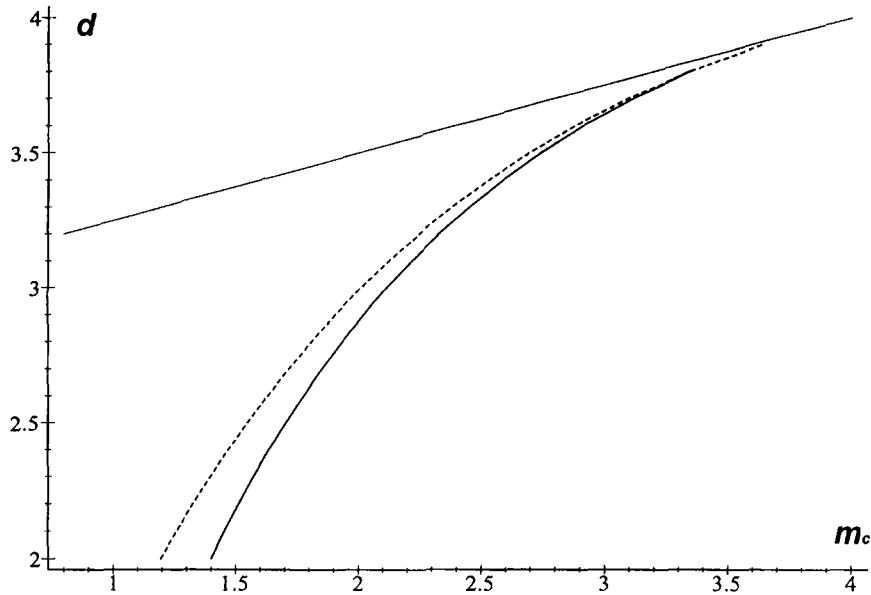


Fig. 6. The dependence of the marginal order parameter component number m_c on the space dimension d . Two- and three-loop results are shown by the dashed and thick solid lines respectively, the ϵ -expansion data $m_c = 4 - 4\epsilon$ are depicted by the thin solid line.

approach to the exact result. One can expect that already the four-loop case will improve the estimates for m_c for all $2 \leq d < 4$. Let us also note that the determination of m_c may serve as a test for improving the resummation scheme.

4. CONCLUSIONS

The goal of this paper is to study the critical behaviour of the weakly diluted quenched Ising model in the case when the space dimension d continuously changes from $d=2$ to $d=4$.

As it was mentioned in the Introduction, the study of the pure Ising model at arbitrary d , which corresponds to a scalar field-theoretical model with one coupling constant, is the subject of a great deal of papers. It is not the case for the model with a more complicated symmetry. In particular, here we study a model with two couplings corresponding to terms of different symmetry in the Lagrangian (1). Such a problem was studied previously on the basis of the scaling-field method,⁽³⁶⁾ and field-theoretical fixed dimension renormalization group calculations within a two-loop level of the perturbation theory are available.⁽⁶⁷⁾

Our calculations hold within the theoretical scheme of refs. 23 and 67. This approach appears to be one amidst other possible calculation schemes for many tasks; however, in our case it seems to have no alternatives within the field-theoretical approach.

Being asymptotic, the resulting series for the RG-functions are to be resummed. In the present study we have chosen the Padé-Borel and the Chisholm-Borel resummation techniques. Restricting ourselves to analytic expressions for the resummed functions, we present numerical data mainly obtained on the basis of the Chisholm-Borel resummation technique. Note that the absence of any information on the high-order behaviour of the obtained series for the RG-functions does not allow one to apply other resummation schemes, e.g. those based on the conformal mapping technique.⁽⁸³⁾

The quantitative description of the critical behaviour of the model is steady from the point of view of passing from the two- to the three-loop approximation. Smaller agreement between the two- and the three-loop approximations at d far away from $d=4$ may be explained in a way that the precision of computing falls down with the increase of the expansion parameter which takes place at decrease of d . The real parts of eigenvalues corresponding to the mixed point seem to remain positive up to $d=4$, which testifies that at arbitrary d the weakly diluted quenched Ising model is described by the mixed fixed point.

APPENDIX

Here we have collected the most lengthy expressions for the three-loop contributions to the RG-functions. The three-loop part of the β_u -function reads:

$$\beta_u^{3LA}(u, v) = \beta_u^{3,0} u^3 + \beta_u^{2,1} u^2 v + \beta_u^{1,2} u v^2 + \beta_u^{0,3} v^3 \quad (26)$$

where

$$\begin{aligned} \beta_u^{3,0} = & -\frac{1}{(m+8)^3} [-4(31m^2 + 430m + 1240) i_1 + (m+8)(m+2) \\ & \times (-(3d+8) i_2 + 12(i_3 + i_8)) + 48(m^2 + 20m + 60) i_4 \\ & + 24(2m^2 + 21m + 58) i_5 + 6(3m^2 + 22m + 56) i_6 \\ & + 24(5m + 22) i_7 + 8(4m^2 + 61m + 178)] \end{aligned}$$

$$\begin{aligned}
\beta_u^{2,1} &= -\frac{2}{(m+8)^2(mn+8)} \left[-12(17m^2 + 256m + 780) i_1 \right. \\
&\quad + (m+2)(-(3dm + 42d + 16m + 80) i_2 + 12(m+14) i_3 \\
&\quad + 18(m+8) i_8) + 24(3m^2 + 70m + 224) i_4 \\
&\quad + 6(15m^2 + 158m + 448) i_5 + 6(3m^2 + 32m + 100) i_6 \\
&\quad \left. + 48(5m + 22) i_7 + 6(9m^2 + 146m + 448) \right] \\
\beta_u^{1,2} &= -\frac{1}{(m+8)(mn+8)^2} \left[-12(19m^2n + 80mn + 470m + 2032) i_1 \right. \\
&\quad - (8(mn+8)(3d+4) + m(3dmn + 40mn + 78d + 176)) i_2 \\
&\quad + 12(m^2n + 8mn + 26m + 64) i_3 + 48(m^2n + 8mn + 68m + 292) i_4 \\
&\quad + 12(11m^2n + 34mn + 136m + 584) i_5 \\
&\quad + 6(m^2n + 8mn + 50m + 256) i_6 + 576(m+5) i_7 \\
&\quad \left. + 36(m+2)(mn+8) i_8 + 12(5m^2n + 22mn + 136m + 584) \right] \\
\beta_u^{0,3} &= -\frac{4}{(mn+8)^3} \left[6(mn+10)(-2(mn+23) i_1 + 3i_6) \right. \\
&\quad + (mn+2)(-(4mn+9d+8) i_2 + 36i_3 + 3(mn+8) i_8) \\
&\quad + 72(3mn+22) i_4 + 9(m^2n^2 + 14mn + 88) i_5 \\
&\quad \left. + 24(mn+14) i_7 + 3(m^2n^2 + 38mn + 264) \right]
\end{aligned}$$

The three-loop part of the β_v -function reads:

$$\beta_v^{3LA}(u, v) = \beta_v^{0,3} v^3 + \beta_v^{1,2} uv^2 = \beta_v^{2,1} u^2 v + \beta_v^{3,0} u^3 \quad (27)$$

where

$$\begin{aligned}
\beta_v^{0,3} &= -\frac{1}{(mn+8)^3} \left[-4(31m^2n^2 + 430mn + 1240) i_1 + (mn+8) \right. \\
&\quad \times (mn+2)(-(3d+8) i_2 + 12(i_3 + i_8)) + 48(m^2n^2 + 20mn + 60) i_4 \\
&\quad + 24(2m^2n^2 + 21mn + 58) i_5 + 6(3m^2n^2 + 22mn + 56) i_6 \\
&\quad \left. + 24(5mn+22) i_7 + 8(4m^2n^2 + 61mn + 178) \right]
\end{aligned}$$

$$\begin{aligned}
\beta_v^{1,2} &= -\frac{4(m+2)}{(mn+8)^2(m+8)} [-4(28mn+275) i_1 \\
&\quad - (3dmn+4mn+15d+56) i_2 + 12(mn+5) i_3 \\
&\quad + 24(2mn+27) i_4 + 3(13mn+100) i_5 + 6(3mn+13) i_6 \\
&\quad + 96i_7 + 9(mn+8) i_8 + (29mn+316)] \\
\beta_v^{2,1} &= -\frac{m+2}{(mn+8)(m+8)^2} [-12(mn+42m+224) i_1 \\
&\quad - (3dmn+12dm-8mn+48d+16m+256) i_2 \\
&\quad + 12(mn+4m+16) i_3 + 48(5m+34) i_4 \\
&\quad + 12(13m+56) i_5 + 6(3mn+14m+40) i_6 \\
&\quad + 144i_7 + 36(m+8) i_8 + 12(11m+64)] \\
\beta_v^{3,0} &= -\frac{2(m+2)}{(m+8)^3} [-4(11m+70) i_1 - 3(dm+2d+16) i_2 \\
&\quad + 6(m+2)(2i_3+3i_6) + 2(m+8)(12i_4+3i_5+3i_8+5)]
\end{aligned}$$

The three-loop part of the γ_ϕ -function reads:

$$\begin{aligned}
\gamma_\phi^{3LA}(u, v) &= -\left[\frac{m+2}{(m+8)^2} u^3 + \frac{3(m+2)}{(m+8)(mn+8)} u^2v \right. \\
&\quad \left. + \frac{3(m+2)}{(m+8)(mn+8)} uv^2 + \frac{mn+2}{(mn+8)^2} v^3 \right] (3i_8 - 4i_2)
\end{aligned}$$

The three-loop part of the $\bar{\gamma}_{\phi^2}$ -function reads:

$$\bar{\gamma}_{\phi^2}(u, v) = \bar{\gamma}_{\phi^2}^{3,0} u^3 + \bar{\gamma}_{\phi^2}^{2,1} u^2v + \bar{\gamma}_{\phi^2}^{1,2} uv^2 + \bar{\gamma}_{\phi^2}^{0,3} v^3 \quad (29)$$

where

$$\begin{aligned}
\bar{\gamma}_{\phi^2}^{3,0} &= \frac{m+2}{(m+8)^2} [-4(11m+70) i_1 + (m+2)(-(3d-8) i_2 \\
&\quad + 12i_3 + 18i_6) + 2(m+8)(12i_4 + 3i_5 + 5)]
\end{aligned}$$

$$\begin{aligned}\bar{\gamma}_{\phi^2}^{2,1} &= \frac{m+2}{(m+8)^2(mn+8)} [-12(mn+10m+70) i_1 \\ &\quad + (mn+2m+6)(-(3d-8) i_2 + 12i_3 + 18i_6) \\ &\quad + 6(m+8)(12i_4 + 3i_5 + 5)] \\ \bar{\gamma}_{\phi^2}^{1,2} &= \frac{3(m+2)}{(m+8)(mn+8)^2} [-4(11mn+70) i_1 \\ &\quad + (mn+2)(-(3d-8) i_2 + 12i_3 + 18i_6) + 2(mn+8)(12i_4 + 3i_5 + 5)] \\ \bar{\gamma}_{\phi^2}^{0,3} &= \frac{mn+2}{(mn+8)^3} [-4(11mn+70) i_1 \\ &\quad + (mn+2)(-(3d-8) i_2 + 12i_3 + 18i_6) + 2(mn+8)(12i_4 + 3i_5 + 5)]\end{aligned}$$

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